Contractively embedded invariant subspaces

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To Joseph A. Ball with gratitude and admiration

Abstract. This paper focuses on representations of contractively embedded invariant subspaces in several variables. We present a version of the de Branges theorem for n-tuples of multiplication operators by the coordinate functions on analytic reproducing kernel Hilbert spaces over the unit ball $\mathbb{B}^n$ and the Hardy space over the unit polydics $\mathbb{D}^n$ in $\mathbb{C}^n$.

Mathematics Subject Classification (2010). Primary 46C07, 46E22, 47A13; Secondary 47A15, 47B32.

Keywords. Invariant subspaces, de Branges-Rovnyak spaces, Hardy space, reproducing kernel Hilbert spaces, multipliers, bounded analytic functions.

1. Introduction

The theory of contractively embedded invariant and co-invariant (not necessarily closed) subspaces for the shift operator on the Hardy space was initiated by L. de Branges. This theory was laid out more systematically in the mid 60's by de Branges and Rovnyak (see the monograph by de Branges and Rovnyak [18]). The de Branges and Rovnyak's approach to the theory of contractively embedded invariant and co-invariant subspaces for shift operators on reproducing kernel Hilbert spaces has proved very fruitful in analysing operator and function theoretic problems. As is well known, it was this theory that led de Branges to the affirmative solution of the Bieberbach conjecture [17].

The first named author's research work is partially supported by an INSPIRE faculty fellowship (IFA-MA-02) funded by DST. The second author is supported in part by (1) National Board of Higher Mathematics (NBHM), India, grant NBHM/R.P.64/2014, and (2) Mathematical Research Impact Centric Support (MATRICS) grant, File No : MTR/2017/000522, by the Science and Engineering Research Board (SERB), Department of Science & Technology (DST), Government of India.

The purpose of this note is to analyze the structure of contractively embedded (not necessarily closed) invariant subspaces for tuples of multiplication operators by the coordinate functions on reproducing kernel Hilbert spaces in several variables. Recall that a Hilbert space $\mathcal{H}$ is said to be contractively embedded in a Hilbert space $\mathcal{K}$ if $\mathcal{H}$ is a vector subspace of $\mathcal{K}$ and the inclusion map $i_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathcal{K}$ is a contraction. Obviously, the latter condition is equivalent to

$$\|f\|_{\mathcal{K}} \leq \|f\|_{\mathcal{H}},$$

for all $f \in \mathcal{H}$, where $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{K}}$ denotes the norms on $\mathcal{H}$ and $\mathcal{K}$, respectively. It follows in particular that a closed subspace of a Hilbert space is contractively (or isometrically, as an embedding) embedded in the larger Hilbert space.

Now let $\mathcal{K}$ be a Hilbert space, and let $\mathcal{H}$ be a Hilbert space that is contractively embedded in $\mathcal{K}$. Let $(T_1, \ldots, T_n)$ be an n-tuple of commuting bounded linear operators on $\mathcal{K}$, that is,

$$T_i T_j = T_j T_i,$$

for all $i, j = 1, \ldots, n$. Let $\mathcal{H}$ be an *invariant subspace* for $(T_1, \ldots, T_n)$, that is,

$$T_i \mathcal{H} \subseteq \mathcal{H},$$

for all $i = 1, \ldots, n$. Suppose that $T_i|_{\mathcal{H}}$ is bounded on $\mathcal{H}$, that is, there exists $M > 0$ such that

$$\|T_i f\|_{\mathcal{H}} \leq M \|f\|_{\mathcal{H}},$$

for all $f \in \mathcal{H}$ and $i = 1, \ldots, n$. Then clearly $(T_1|_{\mathcal{H}}, \ldots, T_n|_{\mathcal{H}})$ is an n-tuple of commuting bounded linear operators on $\mathcal{H}$. The question of interest here is to represent $\mathcal{H}$ in terms of the (algebraic or analytic properties of the) tuple $(T_1, \ldots, T_n)$.

We pause now to examine one concrete example of the above invariant subspace problem. Following standard notation, let $H^2(\mathbb{D})$ denote the Hardy space over the unit disc $\mathbb{D}$. Let M_z on $H^2(\mathbb{D})$ be the multiplication operator by the independent variable z, that is,

$$(M_z f)(w) = w f(w),$$

for all $f \in H^2(\mathbb{D})$ and $w \in \mathbb{D}$. It follows that M_z is a shift of multiplicity one (see Section 3). Let $\mathcal{H}$ be a Hilbert space contractively embedded in $H^2(\mathbb{D})$ such that $M_z \mathcal{H} \subseteq \mathcal{H}$. If $M_z|_{\mathcal{H}}$ is an isometry on $\mathcal{H}$, then the celebrated theorem of de Branges says that there is a function $\varphi \in H^\infty(\mathbb{D})$ such that $\|\varphi\|_\infty \leq 1$ and

$$\mathcal{H} = \varphi H^2(\mathbb{D}).$$

Recall that $H^\infty(\mathbb{D})$ is the Banach algebra of all bounded analytic functions on the unit disc $\mathbb{D}$ equipped with the supremum norm [28]. Here the norm on $\mathcal{H}$ is the range norm induced by the injective multiplier M_φ on $H^2(\mathbb{D})$, that is,

$$\|\varphi f\|_{\mathcal{H}} = \|f\|_{H^2(\mathbb{D})},$$

for all $f \in H^2(\mathbb{D})$ (cf. Section 3 in [34] and Theorems 3.5 and 3.7 in [38]). In this context, we refer the reader to the beautiful survey by Ball and Bolotnikov [5] on de Branges-Rovnyak spaces in both one and several variables, the monographs by Fricain and Mashreghi [19], Sarason [33, 34], Nikolśkiĭ and Vasyunin [29], Sand [31] and Timotin [38]. Also see Singh and Thukral [37] and Sahni and Singh [30]. Another important and relevant piece of work is due to Ball and Kriete [12] and Crofoot [16]. The reader can also see the papers by Chevrot, Guillot and Ransford [14], Costara and Ransford [15] and Sarason [32] in connection with the de Branges-Rovnyak models and (generalized) Dirichlet spaces.

A natural question is now to ask for similar representations of contractively embedded invariant subspaces for (tuples of) multiplication operators by the coordinate function(s) within the framework of analytic reproducing kernel Hilbert spaces [3] in one and several variables.

In Theorems 2.2, 2.3 and 3.3, we present a solution for this problem in the setting of commuting row contractions on Hilbert spaces and analytic Hilbert spaces (see the definition in Section 2) and tuples of shift operators on vector-valued Hardy spaces over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, respectively.

The proofs of Theorems 2.2 and 2.3 involve a careful adaptation of techniques used in [35] and [36]. Whereas the setting and the proof of our invariant subspace theorem for the shift on the Hardy space over the polydisc, Theorem 3.3, is closely related to the recently initiated work [23] on the classification of (closed) invariant subspace problem for the Hardy space in several variables.

A somewhat more intriguing and complex problem is the classification of contractively embedded invariant subspaces which admit a co-invariant complemented subspace. Note that an important aspect of the de Branges-Rovnyak theory is the complementations of invariant subspaces of the Hardy space: A contractively embedded invariant subspace for M_z on $H^2(\mathbb{D})$ is complemented in $H^2(\mathbb{D})$ by an M_z^*-invariant (not necessarily closed) subspace (cf. Subsection 3.4 in [38]). We postpone the general discussion on complemented invariant subspaces for a future paper and refer the reader to the papers by Ball, Bolotnikov and Fang [7, 8, 9], Ball, Bolotnikov and ter Horst [10, 11] and Benhida and Timotin [13] for related results in the setting of Drury-Arveson space [4].

For the remainder, we adapt the following notations: $\boldsymbol{z}$ denotes the element $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$, $z_i \in \mathbb{C}$, $\mathbb{D}^n = \{\boldsymbol{z} \in \mathbb{C}^n : |z_i| < 1, i = 1, \ldots, n\}$, $\mathbb{B}^n = \{\boldsymbol{z} \in \mathbb{C}^n : \|\boldsymbol{z}\|_{\mathbb{C}^n} < 1\}$ and

$$\mathbb{Z}_+^n = \{\boldsymbol{k} = (k_1, \ldots, k_n) : k_i \in \mathbb{Z}_+, i = 1, \ldots, n\}.$$

Also for each multi-index $\boldsymbol{k} \in \mathbb{Z}_+^n$, commuting tuple $T = (T_1, \ldots, T_n)$ on a Hilbert space $\mathcal{H}$, and $\boldsymbol{z} \in \mathbb{C}^n$ we denote

$$T^{\boldsymbol{k}} = T_1^{k_1} \cdots T_n^{k_n} \quad \text{and} \quad \boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}.$$

2. Row contractions and reproducing kernel Hilbert spaces

Let n be a natural number, and let $\mathcal{H}$ be a Hilbert space. A commuting tuple of bounded linear operators $(T_1, \dots, T_n)$ acting on $\mathcal{H}$ is called a *row contraction* if the row operator $(T_1, \dots, T_n) : \mathcal{H}^n \to \mathcal{H}$ defined by

$$(T_1, \dots, T_n) \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = T_1 h_1 + \dots + T_n h_n,$$

for all $h_1, \dots, h_n \in \mathcal{H}$, is a contraction. Evidently, the tuple $(T_1, \dots, T_n)$ is a row contraction if and only if

$$\|T_1 h_1 + \dots + T_n h_n\|^2 \leq \|h_1\|^2 + \dots + \|h_n\|^2,$$

for all $h_1, \dots, h_n \in \mathcal{H}$, or equivalently if

$$\sum_{i=1}^n T_i T_i^* \leq I_{\mathcal{H}}.$$

For a row contraction $T = (T_1, \dots, T_n)$ on a Hilbert space $\mathcal{H}$, we define the *defect operator* and the *defect space* of T as

$$D_T = (I_{\mathcal{H}} - \sum_{i=1}^n T_i T_i^*)^{\frac{1}{2}},$$

and

$$\mathcal{D}_T = \overline{\operatorname{ran}} D_T$$

respectively. Consider the map $P_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$P_T(X) = \sum_{i=1}^n T_i X T_i^*,$$

for all $X \in \mathcal{B}(\mathcal{H})$. Clearly, P_T is a completely positive map. Moreover, since

$$I_{\mathcal{H}} \geq P_T(I_{\mathcal{H}}) \geq P_T^2(I_{\mathcal{H}}) \geq \dots \geq 0,$$

it follows that

$$P_\infty(T) = \text{SOT} - \lim_{m \to \infty} P_T^m(I_{\mathcal{H}}),$$

exists and $0 \leq P_\infty(T) \leq I_{\mathcal{H}}$. We say that T is a *pure row contraction* if

$$P_\infty(T) = 0.$$

Standard examples of pure row contractions are the multiplication operator tuples by the coordinate functions on the Drury-Arveson space, the Hardy space, the Bergman space and the weighted Bergman spaces over $\mathbb{B}^n$. In fact, for each $\lambda \geq 1$, the multiplication operator tuple $(M_{z_1}, \dots, M_{z_n})$ on the reproducing kernel Hilbert space $\mathcal{H}_{K_\lambda}$ is a pure row contraction, where

$$K_\lambda(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-\lambda}, \tag{2.1}$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$ (cf. Proposition 4.1 in [35]). Note that the Drury-Arveson space H^2_n, the Hardy space $H^2(\mathbb{B}^n)$, the Bergman space $L^2_a(\mathbb{B}^n)$, and the weighted Bergman space $L^2_{a,\alpha}(\mathbb{B}^n)$, with $\alpha > 0$, are reproducing kernel Hilbert spaces with kernel K_λ for $\lambda = 1, n$, $n+1$ and $n+1+\alpha$, respectively.

Let $\mathcal{E}$ be a Hilbert space. We identify the Hilbert tensor product $H^2_n \otimes \mathcal{E}$ with the $\mathcal{E}$-valued Drury-Arveson space $H^2_n(\mathcal{E})$, or the $\mathcal{E}$-valued reproducing kernel Hilbert space with kernel function

$$\mathbb{B}^n \times \mathbb{B}^n \ni (\boldsymbol{z}, \boldsymbol{w}) \mapsto (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-1} I_{\mathcal{E}}.$$

Then

$$H^2_n(\mathcal{E}) = \{f \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : f(z) = \sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} a_{\boldsymbol{k}} z^{\boldsymbol{k}}, a_{\boldsymbol{k}} \in \mathcal{E}, \|f\|^2 := \sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} \frac{\|a_{\boldsymbol{k}}\|^2}{\gamma_{\boldsymbol{k}}} < \infty\},$$

where $\gamma_{\boldsymbol{k}} = \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}$ are the multinomial coefficients, $\boldsymbol{k} \in \mathbb{Z}^n_+$ (cf. [4] and [22]).

Now let $\mathcal{K}$ be a Hilbert space, and let $\mathcal{H}$ be a Hilbert space that is contractively embedded in $\mathcal{K}$. Let $T = (T_1, \ldots, T_n)$ be a pure row contraction on $\mathcal{K}$. Let

$$T_j \mathcal{H} \subseteq \mathcal{H},$$

and let

$$R_j = T_j|_{\mathcal{H}},$$

for all $j = 1, \ldots, n$. Suppose that $R = (R_1, \ldots, R_n)$ is a row contraction on $\mathcal{H}$, that is,

$$\|\sum_{i=1}^n R_i h_i\|^2_{\mathcal{H}} = \|\sum_{i=1}^n T_i h_i\|^2_{\mathcal{H}} \leq \sum_{i=1}^n \|h_i\|^2_{\mathcal{H}},$$

for all $h_1, \ldots, h_n \in \mathcal{H}$. First we claim that $(R_1, \ldots, R_n)$ is a pure row contraction. Indeed, observe that

$$i_{\mathcal{H}} R_j = T_j i_{\mathcal{H}},$$

for all $j = 1, \ldots, n$. Then

$$i_{\mathcal{H}} R^{\boldsymbol{k}} = T^{\boldsymbol{k}} i_{\mathcal{H}},$$

and so

$$R^{*\boldsymbol{k}} i^*_{\mathcal{H}} = i^*_{\mathcal{H}} T^{*\boldsymbol{k}},$$

for all $\boldsymbol{k} \in \mathbb{Z}^n_+$. This yields

$$i_{\mathcal{H}} R^{\boldsymbol{k}} R^{*\boldsymbol{k}} i^*_{\mathcal{H}} = T^{\boldsymbol{k}} i_{\mathcal{H}} i^*_{\mathcal{H}} T^{*\boldsymbol{k}},$$

for all $\boldsymbol{k} \in \mathbb{Z}^n_+$, and hence

$$i_{\mathcal{H}} P^m_R(I_{\mathcal{H}}) i^*_{\mathcal{H}} = P^m_T(i_{\mathcal{H}} i^*_{\mathcal{H}}),$$

for each $m \geq 0$. Since $P^m_T : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ is a (completely) positive map and

$$i_{\mathcal{H}} i^*_{\mathcal{H}} \leq I_{\mathcal{K}},$$

(recall that $i_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathcal{K}$ is a contraction) we obtain that

$$P_T^m(i_{\mathcal{H}} i_{\mathcal{H}}^*) \leq P_T^m(I_{\mathcal{K}}),$$

and hence

$$i_{\mathcal{H}} P_R^m(I_{\mathcal{H}}) i_{\mathcal{H}}^* \leq P_T^m(I_{\mathcal{K}}),$$

for all $m \geq 0$. Now for $f \in \mathcal{K}$ and $m \geq 0$, we compute

$$\begin{aligned} \|P_R^m(I_{\mathcal{H}})^{\frac{1}{2}} i_{\mathcal{H}}^* f\|_{\mathcal{H}}^2 &= \langle P_R^m(I_{\mathcal{H}}) i_{\mathcal{H}}^* f, i_{\mathcal{H}}^* f\rangle_{\mathcal{H}} \\ &= \langle i_{\mathcal{H}} P_R^m(I_{\mathcal{H}}) i_{\mathcal{H}}^* f, f\rangle_{\mathcal{K}} \\ &\leq \langle P_T^m(I_{\mathcal{K}}) f, f\rangle_{\mathcal{K}}. \end{aligned}$$

Since $(T_1, \ldots, T_n)$ is a pure row contraction we see that

$$\lim_{m\to\infty} \|P_R^m(I_{\mathcal{H}})^{\frac{1}{2}} i_{\mathcal{H}}^* f\|_{\mathcal{H}} = 0,$$

for all $f \in \mathcal{K}$. On the other hand, since $i_{\mathcal{H}}$ is one-to-one we see that $i_{\mathcal{H}}^* : \mathcal{K} \to \mathcal{H}$ has dense range, and hence by continuity

$$SOT - \lim_{m\to\infty} P_R^m(I_{\mathcal{H}})^{\frac{1}{2}} = 0.$$

Since the sequence of positive operators $\{P_R^m(I_{\mathcal{H}})\}_{m\geq 0}$ is uniformly bounded (by $\|I_{\mathcal{H}}\| = 1$) we obtain that

$$SOT - \lim_{m\to\infty} P_R^m(I_{\mathcal{H}}) = 0,$$

that is, $(R_1, \ldots, R_n)$ on $\mathcal{H}$ is a pure row contraction.

At this point we pause to recall the dilation result due to Jewell and Lubin [22] and Muller and Vasilescu [27] (also see Arveson [4]) which says that a pure row contraction is jointly unitarily equivalent to the compression of the tuple of multiplication operators by the coordinate functions $\{z_1, \ldots, z_n\}$ on a vector-valued Drury-Arveson space to a joint co-invariant subspace. In other words, the multiplication operator tuple $(M_{z_1}, \ldots, M_{z_n})$ on the Drury-Arveson space plays the role of the model pure row contraction. We state this more formally as follows (see Theorem 3.1 [35] for a proof):

Theorem 2.1. *Let $\mathcal{L}$ be a Hilbert space, and let $X = (X_1, \ldots, X_n)$ be a pure row contraction on $\mathcal{L}$. Then there exists a co-isometry $\Pi_X : H_n^2(\mathcal{D}_X) \to \mathcal{L}$ such that*

$$\Pi_X M_{z_j} = X_j \Pi_X,$$

for all $j = 1, \ldots, n$.

Therefore, by the above dilation theorem applied to the pure row contraction $(R_1, \ldots, R_n)$, we get a co-isometry $\Pi_R : H_n^2(\mathcal{D}_R) \to \mathcal{H}$ such that

$$\Pi_R M_{z_j} = R_j \Pi_R,$$

for all $j = 1, \ldots, n$. Let

$$\Pi = i_{\mathcal{H}} \circ \Pi_R.$$

It follows that $\Pi : H_n^2(\mathcal{D}_R) \to \mathcal{K}$ is a contraction and

$$\text{ran}\, \Pi = \mathcal{H}.$$

Moreover, since $i_{\mathcal{H}} R_j = T_j i_{\mathcal{H}}$, we have that

$$\Pi M_{z_j} = T_j \Pi,$$

for all $j = 1, \ldots, n$. We summarize these results as follows:

THEOREM 2.2. *Let $\mathcal{K}$ be a Hilbert space, and let $(T_1, \ldots, T_n)$ be a pure row contraction on $\mathcal{K}$. Let $\mathcal{H}$ be a Hilbert space that is contractively embedded in $\mathcal{K}$. Let $T_j \mathcal{H} \subseteq \mathcal{H}$ and*

$$R_j = T_j|_{\mathcal{H}},$$

for all $j = 1, \ldots, n$. Let $(R_1, \ldots, R_n)$ be a row contraction on $\mathcal{H}$. Then $(R_1, \ldots, R_n)$ is a pure row contraction and there exist a Hilbert space $\mathcal{E}_*$ and a contraction $\Pi : H^2_n(\mathcal{E}_*) \to \mathcal{K}$ such that

$$\Pi M_{z_j} = T_j \Pi,$$

for all $j = 1, \ldots, n$, and

$$\textit{ran}\, \Pi = \mathcal{H}.$$

Of particular interest is the case where $(T_1, \ldots, T_n)$ is the n-tuple of multiplication operators on a Hilbert space of analytic functions in the unit ball. To this end, we first need to introduce analytic Hilbert spaces over $\mathbb{B}^n$ (see [35] and [36] for more details).

Let $K : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be a positive definite kernel such that $K(\boldsymbol{z}, \boldsymbol{w})$ is holomorphic in the $\{z_1, \ldots, z_n\}$ variables and anti-holomorphic in $\{w_1, \ldots, w_n\}$ variables. Then the corresponding reproducing kernel Hilbert space $\mathcal{H}_K$ is a Hilbert space of holomorphic functions in $\mathbb{B}^n$. We say that $\mathcal{H}_K$ is an *analytic Hilbert space* if $(M_{z_1}, \ldots, M_{z_n})$, the n-tuple of multiplication operators by the coordinate functions $\{z_1, \ldots, z_n\}$, defines a pure row contraction on $\mathcal{H}_K$. In other words, M_{z_j} on $\mathcal{H}_K$ defined by

$$(M_{z_j} f)(\boldsymbol{w}) = w_j f(\boldsymbol{w}) \qquad (f \in \mathcal{H}_K, \boldsymbol{w} \in \mathbb{B}^n),$$

is bounded for all $j = 1, \ldots, n$, the commuting tuple $M_z = (M_{z_1}, \ldots, M_{z_n})$ on $\mathcal{H}_K$ satisfies the positivity condition

$$\sum_{i=1}^{n} M_{z_i} M_{z_i}^* \leq I_{\mathcal{H}_K},$$

and

$$P_\infty(M_z) = 0.$$

Let $\mathcal{E}$ be a Hilbert space. Consider the $\mathcal{E}$-valued reproducing kernel Hilbert space $\mathcal{H}_{K_\lambda} \otimes \mathcal{E}$, $\lambda \geq 1$, where K_λ is defined as in (2.1). Then the reproducing kernel Hilbert space $\mathcal{H}_{K_\lambda} \otimes \mathcal{E}$ is analytic, as is well-known and also follows, for example, from Proposition 4.1 in [35]. In particular, the vector-valued Drury-Arveson space $H^2_n \otimes \mathcal{E}$, the Hardy space $H^2(\mathbb{B}^n) \otimes \mathcal{E}$, the Bergman space $L^2_a(\mathbb{B}^n) \otimes \mathcal{E}$, and the vector-valued weighted Bergman spaces $L^2_{a,\alpha}(\mathbb{B}^n) \otimes \mathcal{E}$, with $\alpha > 0$, are analytic Hilbert spaces.

We finally recall a characterization of intertwining maps between vector-valued Drury-Arveson space and analytic Hilbert spaces (cf. Proposition 4.2

in [35]). Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, $\mathcal{H}_K$ be an analytic Hilbert space and let $X \in \mathcal{B}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_K \otimes \mathcal{E}_2)$. Then

$$X(M_{z_i} \otimes I_{\mathcal{E}_1}) = (M_{z_i} \otimes I_{\mathcal{E}_2})X,$$

for all $i = 1, \ldots, n$, if and only if there exists a multipler $\Theta \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_K \otimes \mathcal{E}_2)$ such that

$$X = M_\Theta. \tag{2.2}$$

Recall that the multiplier space $\mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_K \otimes \mathcal{E}_2)$ is the Banach space of all operator-valued analytic functions $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ such that

$$\Theta f \in \mathcal{H}_K \otimes \mathcal{E}_2,$$

for all $f \in H^2_n \otimes \mathcal{E}_1$. Note that if $\Theta \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_K \otimes \mathcal{E}_2)$, then the multiplication operator M_Θ defined by

$$(M_\Theta f)(\boldsymbol{w}) = \Theta(\boldsymbol{w}) f(\boldsymbol{w}),$$

for all $f \in H^2_n \otimes \mathcal{E}_1$ and $\boldsymbol{w} \in \mathbb{B}^n$, is a bounded linear operator (by the closed graph theorem) from $H^2_n \otimes \mathcal{E}_1$ to $\mathcal{H}_K \otimes \mathcal{E}_2$ (cf. [20], [26] and [35]). The next corollary now follows directly from Theorem 2.2.

Theorem 2.3. *Let $\mathcal{E}_*$ be a Hilbert space, and let $\mathcal{H}_K$ be an analytic Hilbert space. Let $\mathcal{S}$ be a Hilbert space that is contractively embedded in $\mathcal{H}_K \otimes \mathcal{E}_*$. Let $M_{z_j}\mathcal{S} \subseteq \mathcal{S}$ and*

$$R_j = M_{z_j}|_{\mathcal{S}},$$

for all $j = 1, \ldots, n$, and suppose that $(R_1, \ldots, R_n)$ is a row contraction on $\mathcal{S}$. Then $(R_1, \ldots, R_n)$ is a pure row contraction and there exist a Hilbert space $\mathcal{E}$ and a contractive multiplier $\Theta \in \mathcal{M}(H^2_n \otimes \mathcal{E}, \mathcal{H}_K \otimes \mathcal{E}_*)$ such that

$$\mathcal{S} = \Theta H^2_n(\mathcal{E}).$$

In the case when $\mathcal{H}_K$ is the Drury-Arveson space H^2_n, see the early results in Benhida and Timotin (Theorem 4.2 [13]). In this context we also refer to McCullough and Trent [26] and Greene, Richter and Sundberg [20].

We would like to point out that the theory of contractively embedded backward shift invariant subspaces in reproducing kernel Hilbert spaces and the de Branges-Rovnyak models, in the setting of row contractions, are closely related to the Gleason's problem [1]. In this context, the reader should consult the papers by Alpay and Dubi [2], Ball and Bolotnikov [6], Ball, Bolotnikov and Fang [7, 9], Ball, Bolotnikov and ter Horst [10, 11], Benhida and Timotin [13] and Martin and Ramanantoanina [25].

3. Hardy space over the polydisc

Let n be a natural number. Given a Hilbert space $\mathcal{E}$, we denote by $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ the $\mathcal{E}$-valued Hardy space over the polydisc $\mathbb{D}^{n+1}$. In this section we aim to analyze the structure of contractively embedded invariant subspaces for the multiplication tuple on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$. The principle of our method is based on the idea [23] that one can represent the tuple of shifts on the Hardy space over $\mathbb{D}^{n+1}$ by a natural $(n+1)$-tuple of multiplication operators on a vector-valued Hardy space over the unit disc. This is the main content of the following theorem (see Theorem 3.1 in [23]).

THEOREM 3.1. *Let n be a natural number, and let $\mathcal{E}$ be a Hilbert space. Let*

$$\mathcal{E}_n = H^2_{\mathcal{E}}(\mathbb{D}^n).$$

For each $i = 1, \ldots, n$, let $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{E}_n)}(\mathbb{D})$ denote the $\mathcal{B}(\mathcal{E}_n)$-valued constant function on $\mathbb{D}$ defined by

$$\kappa_i(w) = M_{z_i} \in \mathcal{B}(\mathcal{E}_n),$$

for all $w \in \mathbb{D}$, and let M_{κ_i} denote the multiplication operator on $H^2_{\mathcal{E}_n}(\mathbb{D})$ defined by

$$M_{\kappa_i} f = \kappa_i f,$$

for all $f \in H^2_{\mathcal{E}_n}(\mathbb{D})$. Then $(M_{z_1}, M_{z_2} \ldots, M_{z_{n+1}})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ are unitarily equivalent.

Proof. We briefly sketch only the main ideas behind the proof and refer the reader to the proof of Theorem 3.1 in [23] for details. Since the linear spans of

$$\{z_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}} \eta : k_1, \ldots, k_{n+1} \geq 0, \eta \in \mathcal{E}\} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^{n+1}),$$

and

$$\{z^k (z_1^{k_1} \cdots z_n^{k_n} \eta) : k, k_1, \ldots, k_n \geq 0, \eta \in \mathcal{E}\} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D}),$$

are dense in $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ and $H^2_{\mathcal{E}_n}(\mathbb{D})$, respectively, it follows that the map $U : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ defined by

$$U(z_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}} \eta) = z^{k_1} (z_1^{k_2} \cdots z_n^{k_{n+1}} \eta),$$

for all $k_1, \ldots, k_{n+1} \geq 0$ and $\eta \in \mathcal{E}$, is a unitary operator. Clearly

$$U M_{z_1} = M_z U,$$

and an easy computation yields

$$U M_{z_i} = M_{\kappa_{i-1}} U,$$

for all $i = 2, \ldots, n$. This completes the proof. □

In view of the above theorem, we can now consider the problem of contractively embedded invariant subspaces for the tuple $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ instead of the tuple of multiplication operators $(M_{z_1}, M_{z_2}, \ldots, M_{z_{n+1}})$ on the vector-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$.

Before we proceed to the main result of this section, we need one more result concerning representations of commutators of shift [21] operators. Here our approach follows that of [23] and [24]. Recall that an isometry V on a Hilbert space $\mathcal{H}$ is said to be a *shift* if

$$SOT - \lim_{m \to \infty} V^{*m} = 0,$$

that is, $\|V^{*m}f\| \to 0$ as $m \to \infty$ for all $f \in \mathcal{H}$, or equivalently, if there is no non trivial reducing subspace of $\mathcal{H}$ on which V is unitary. Now, if V is a shift on $\mathcal{H}$, then

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} V^m \mathcal{W},$$

where $\mathcal{W} = \ker V^* = \mathcal{H} \ominus V\mathcal{H}$ is the wandering subspace for V. By the above decomposition of $\mathcal{H}$, we see that the map $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ defined by

$$\Pi_V(V^m \eta) = z^m \eta,$$

for all $m \geq 0$ and $\eta \in \mathcal{W}$, is a unitary operator and

$$\Pi_V V = M_z \Pi_V.$$

Following Wold and von Neumann, we call Π_V the *Wold-von Neumann decomposition* of the shift V (see [23] and [24]).

This point of view is very useful in representing the commutators of shifts (see Theorem 2.1 in [24] and Theorem 2.1 in [23]):

THEOREM 3.2. *(Theorem 2.1 in* [24]*) Let $\mathcal{H}$ be a Hilbert space. Let V be a shift on $\mathcal{H}$, and let C be a bounded operator on $\mathcal{H}$. Let Π_V be the Wold-von Neumann decomposition of V, $M = \Pi_V C \Pi_V^*$, and let*

$$\Theta(w) = P_{\mathcal{W}}(I_{\mathcal{H}} - wV^*)^{-1} C|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$. Then $CV = VC$ if and only if $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ and

$$M = M_\Theta.$$

Since $\|wV^*\| = |w|\|V\| < 1$ for all $w \in \mathbb{D}$, it follows that, given a bounded operator C on $\mathcal{W}$, the function Θ as defined above is a $\mathcal{B}(\mathcal{W})$-valued analytic function on $\mathbb{D}$. It is however not clear that Θ is a bounded function on $\mathbb{D}$, that is, $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$. The above theorem says that this is so if and only if C is in the commutator of V.

Proof of Theorem 3.2. Again we will only sketch the proof and refer the reader to Theorem 2.1 in [24] for a more rigorous proof. Certainly, the sufficient part follows from the representation of C (as $\Pi_V^* M_\Theta \Pi_V = C$) and the fact that $M_z M_\Theta = M_\Theta M_z$. The proof for the necessary part relies on the fact that (cf. [24])

$$I_{\mathcal{H}} = \sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m},$$

in the strong operator topology. Indeed, if $CV = VC$, then $MM_z = M_z M$, and so

$$M = M_\Theta,$$

for some bounded analytic function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ (see, for instance, the equality in (2.2)). Let $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$. Then

$$\Theta(w)\eta = (M_\Theta \eta)(w) = (\Pi_V C \Pi_V^* \eta)(w).$$

Since $\Pi_V^*\eta = \eta$ and

$$C\eta = \sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m} C\eta,$$

it follows that

$$\begin{aligned}\Theta(w)\eta &= (\Pi_V C\eta)(w)\\ &= (\Pi_V(\sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m} C\eta))(w)\\ &= (\sum_{m=0}^{\infty} M_z^m (P_{\mathcal{W}} V^{*m} C\eta))(w).\end{aligned}$$

Finally, note that $P_{\mathcal{W}} V^{*m} C\eta \in \mathcal{W}$ for all $m \geq 0$, and hence

$$\Theta(w)\eta = \sum_{m=0}^{\infty} w^m (P_{\mathcal{W}} V^{*m} C\eta),$$

from which the result follows. □

We are now ready for the main result concerning contractively embedded invariant subspaces of vector-valued Hardy spaces.

Let n be a natural number, and let $\mathcal{E}$ be a Hilbert space. Let $\mathcal{S}$ be a Hilbert space that is contractively embedded in $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let

$$z\mathcal{S} \subseteq \mathcal{S},$$

and

$$\kappa_i \mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. Assume that $(R, R_1, \ldots, R_n)$ is an $(n+1)$-tuple of isometries on $\mathcal{S}$, where

$$R = M_z|_{\mathcal{S}},$$

and

$$R_i = M_{\kappa_i}|_{\mathcal{S}},$$

for all $i = 1, \ldots, n$. We have

$$\bigcap_{m=0}^{\infty} R^m \mathcal{S} = \bigcap_{m=0}^{\infty} z^m \mathcal{S} \subseteq \bigcap_{m=0}^{\infty} z^m H^2_{\mathcal{E}_n}(\mathbb{D}).$$

But M_z on $H^2_{\mathcal{E}_n}(\mathbb{D})$ is a pure isometry (shift), that is,

$$\bigcap_{m=0}^{\infty} z^m H^2_{\mathcal{E}_n}(\mathbb{D}) = \{0\},$$

and so it follows that

$$\bigcap_{m=0}^{\infty} R^m \mathcal{S} = \{0\}.$$

Further, since M_{κ_j} is a shift, it follows that

$$\bigcap_{m=0}^{\infty} \kappa_j^m H^2_{\mathcal{E}_n}(\mathbb{D}) = \{0\},$$

and so

$$\bigcap_{m=0}^{\infty} R_j^m \mathcal{S} = \{0\},$$

for all $j = 1, \ldots, n$, follows in a similar way. In other words, $(R, R_1, \ldots, R_n)$ is an $(n+1)$-tuple of commuting shifts on $\mathcal{S}$. Now we argue essentially as in the proof of Theorem 3.2 in [23]. Let

$$\mathcal{W} = \mathcal{S} \ominus z\mathcal{S},$$

and let $\Pi_R : \mathcal{S} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of R on $\mathcal{S}$. In particular, we have

$$R\Pi_R^* = \Pi_R^* M_z. \tag{3.1}$$

Moreover, since $RR_j = R_j R$, applying Theorem 3.2, we have

$$\Pi_R R_j = M_{\Phi_j} \Pi_R, \tag{3.2}$$

where the $\mathcal{B}(\mathcal{W})$-valued analytic function defined by

$$\Phi_j(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - P_{\mathcal{S}} M_z^*)^{-1} M_{\kappa_j}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$, is in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ and $j = 1, \ldots, n$. Now consider the (contractive) inclusion map $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow H^2_{\mathcal{E}_n}(\mathbb{D})$. Set

$$\Pi = i_{\mathcal{S}} \circ \Pi_R^*.$$

Then $\Pi : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ is a contraction. Moreover, since

$$i_{\mathcal{S}} R = M_z i_{\mathcal{S}},$$

and

$$i_{\mathcal{S}} R_j = M_{\kappa_j} i_{\mathcal{S}},$$

it follows from (3.1) and (3.2) that

$$\Pi M_z = M_z \Pi, \tag{3.3}$$

and

$$\Pi M_{\Phi_j} = M_{\kappa_j} \Pi, \tag{3.4}$$

for all $j = 1, \ldots, n$. Then using (2.2), one sees that

$$\Pi = M_\Theta,$$

for some contractive multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E}_n)}(\mathbb{D})$, from (3.3), and hence

$$\Theta \Phi_j = \kappa_j \Theta,$$

from (3.4), for all $j = 1, \ldots, n$. Since ran $i_{\mathcal{S}} = \mathcal{S}$, it follows from the definition of Π that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}).$$

We can therefore state the following analogue of the de Branges theorem in the setting of Hardy space over the unit polydisc:

THEOREM 3.3. *Let n be a natural number, and let $\mathcal{E}$ be a Hilbert space. Let $\mathcal{S}$ be a Hilbert space that is contractively embedded in $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $z\mathcal{S} \subseteq \mathcal{S}$ and*

$$R = M_z|_{\mathcal{S}}.$$

For each $j = 1, \ldots, n$, let $\kappa_j \mathcal{S} \subseteq \mathcal{S}$ and

$$R_j = M_{\kappa_j}|_{\mathcal{S}}.$$

Set $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ *and*

$$\Phi_j(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1}M_{\kappa_j}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ *and* $j = 1, \ldots, n$*. If* $(R, R_1, \ldots, R_n)$ *is an* $(n+1)$*-tuple of commuting isometries on* $\mathcal{S}$*, then* $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ *is an* n*-tuple of commuting shifts on* $H^2_{\mathcal{W}}(\mathbb{D})$ *and there exists a contractive multiplier* $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ *such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

and

$$\kappa_j \Theta = \Theta \Phi_j,$$

for all $j = 1, \ldots, n$*.*

The preceding result, in view of the de Branges and Rovnyak theory, suggests a very interesting question: How can one characterize those contractively embedded invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ which are complemented by invariant subspaces for $(M_z^*, M_{\kappa_1}^*, \ldots, M_{\kappa_n}^*)$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$? The answer to this question is not known.

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